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Constant gravitational fields and redshift of light

V. Patrangenaru

Department of Mathematics, Indiana University, Bloomington, IN 47405-5701, USA

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Abstract

The solutions of Einsteins's equations in a constant energy–momentum tensor field are Ricci curvature homogeneous. Convenient perturbations of a Lorentz solvmanifold yield such curvature homogeneous metrics, prescribing redshift of light and singularities.

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1. Introduction

Ever since Einstein discovered the usefulness of Lorentz geometry in the study of gravitation, the main tool on to describing how mass "curves" the geometry of the ambient space–time $(\mathbf{M}, \mathbf{g})$, and how curvature "moves" matter subject to free fall [10,16], has been the famous equation

$$\mathbf{Ric} - \tfrac{1}{2}\mathbf{S}\mathbf{g} = 8\pi\mathbf{T}. \tag{1.1}$$

Whenever one wishes to find a geometric model for some particular problem in gravitation, one has to solve Eq. (1.1) for $\mathbf{g}$. Usually the stress-energy tensor $\mathbf{T}$ is measured w.r.t. some local orthoframe field Θ (rods and clocks).

Assume for simplicity that the components of $\mathbf{T}$ relative to Θ are constant. Such space–times were already considered in modern cosmology [2]; this is a very likely model for space–time regions which are away from significant gravitational sources, and will be of real interest in remote travels.

Viewed as an equation of sections in $\mathfrak{T}^1_1\mathbf{M}$, (1.1) reads

$$\mathbf{Ric} - \tfrac{1}{2}\mathbf{S}\,\mathrm{Id} = 8\pi\mathbf{T}, \tag{1.2}$$

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where $\mathbf{T} \in C^{\infty}(\mathfrak{T}^1_1\mathbf{M})$, has constant components w.r.t. $\Theta \otimes \Theta^*$, and in particular $\mathrm{Tr}\,\mathbf{T}$ is constant. Then, by taking the trace of (1.2), the scalar curvature $\mathbf{S}$ is constant on $\mathbf{M}$, and the components of the Ricci curvature **Ric** are constant as well.

Definition 1.1. A pseudo-Riemannian manifold, whose Ricci curvature tensor field has constant components w.r.t. some orthoframe field is said to be Ricci curvature homogeneous (R.c.h.).

It is obvious that homogeneous spaces such as Gödel's Universe are R.c.h.'s. A large class of R.c.h.'s are the *curvature homogeneous manifolds (c.h.)*, which are defined similarly, where the Ricci tensor is replaced by the curvature tensor. In dimension 3, there is no distinction between c.h. and R.c.h. manifolds; they and their homogeneous models have been recently studied in [3,7,13,18].

In dimension 4 and above, another interesting class of R.c.h.'s are the *Einstein manifolds (E.m.)*; these are models for a physical vacuum or in other terms, manifolds for which there exists a function $\mathbf{f}$ such that $\mathbf{Ric} = \mathbf{fg}$ [16, p. 96].

Since the property of being a R.c.h. is closed under product, it turns out that there are examples of R.c.h.'s, falling in one and only one or none of the classes c.h.'s or E.m.'s.

However, given that in dimension 3, there are no E.m.'s of nonconstant curvature, in dimension 4, it is more difficult to produce examples of R.c.h.'s which are neither E.m.'s nor c.h.'s.

In this paper we restrict our attention to some c.h. models only, obtained by a perturbation starting from a Lorentz solvmanifold. The idea of perturbing a "nice" metric to obtain solutions of Einstein equations is not new [21,23], still it has never been used before in the covariant setup of [12].

In general, unlike in the homogeneous case [17], the set of nonisometric metrics of given curvature tensor is quite large, depending on functions, and we will provide such an example in Section 3.

A c.h. manifold $(\mathbf{M}, \mathbf{g})$ with the same curvature tensor as a homogeneous manifold $(\mathbf{M}, \mathbf{g}_0)$ is said to have a *homogeneous model*. The model $(\mathbf{M}, \mathbf{g}_0)$ is in general not unique, even if it is symmetric.

We have the following local hierarchy:

constant curvature	R isotropic
locally symmetric	R autoparallel
locally homogeneous	M locally self equivalent
curvature homogeneous	R punctually self equivalent
Ricci curvature homogeneous	Ric punctually self equivalent
constant scalar curvature	S constant
arbitrary	

In the above chart R, (Ric, S) represent the Riemannian (Ricci, scalar) curvature.

A first class of nonhomogeneous Lorentz manifolds modeled on a symmetric space was exhibited in [4].

In this paper, we give other examples in this direction. In order to obtain these new classes of spaces we proceed in two steps: first we will find some highly homogeneous models, using our $\mathfrak{g}$-triple technique [17]. Secondly we deform some flat Lorentz solvmanifolds to curvature homogeneous ones, and determine, in some cases, their homogeneous models.

In spite of their rich geometric structure, from the physical point of view homogeneous Lorentz manifolds are "static" [20]. This is the main reason why R.c.h. space–times are more realistic models, since they predict redshifts of light-like geodesics and singularities suggested by experimental data.

The paper is structured as follows.

Section 2 is concerned with the list of conjugacy classes of Lie subalgebras of the Lorentz algebra $\mathfrak{o}_1(4)$, an ingredient required by the $\mathfrak{g}$-triple method in dimension 4. This list is important in itself, in special relativity, given that each Lie subgroup of the Lorentz group yields a conservation law.

An application of this method to four-dimensional Lorentz homogeneous geometry is presented in Section 3. One shows that Lorentz four-dimensional homogeneous manifolds, whose isotropy algebra is at least two-dimensional, have a symmetric model.

In Section 4, we will deform some flat invariant Lorentz metrics to curvature homogeneous metrics. For some of them we will specify the homogeneous model. Then we will show that for a suitable class of curvature homogeneous manifolds, the light-like particles have cosmological redshifts, or singularities and thus are not static.

2. Constant electromagnetic fields, semispinors and subalgebras of $\mathfrak{o}_1(4)$

To start with, we will determine the highest-dimensional subalgebras of the Lorentz algebra. Let $\| \ \|_1^2$ be the "square of the norm" in the n-dimensional Minkowski vector space $\mathbb{R}_1^n$. An *isotropic* hyperplane is a tangent hyperplane to the *null conus* $\{x \in \mathbb{R}^n, \|x\|_1^2 = 0\}$.

Let $h(n)$ be the highest dimension of a proper subalgebra of the Lorentz algebra $\mathfrak{o}_1(n)$.

As a vector space, $\mathfrak{o}_1(n)$ has the basis $(f_i^j)_{1\le i<j\le n}$,

$$f_i^j = E_i^j - {}_1\delta_{ij} E_j^i, \tag{2.1}$$

where $({}_1\delta_{ij})$ is a diagonal matrix, and all the nonzero entries are 1, except for that from the right lower corner, which is -1, and E_i^j is the usual basis of $\mathfrak{gl}(n)$ [11, Vol. 1, p. 118].

Proposition 2.1. For any $n \in \mathbb{N}$, $5 \neq n > 2$, $h(n) = \frac{1}{2}(n-1)(n-2)+1$. Any $h(n)$-*dimensional Lie subalgebra* of $\mathfrak{o}_1(n)$ is conjugated to the Lie algebra of the group of Lorentz transformations of the plane, that leave invariant an isotropic plane.

Proof. Let $(x^i)_{i=\overline{1,n}}$ be orthogonal coordinates in the Minkowski space, and let $\mathfrak{m}(n)$, $\mathfrak{o}(n-1)$, $\mathfrak{o}(n-2)$ be the Lie algebras of the subgroups of $O_1(n)$, that leave invariant the isotropic hyperplane $x^{n-1} - x^n = 0$, the hyperplane $x^n = 0$, and the subspace $x^{n-1} = x^n = 0$, respectively. Then $\mathfrak{m}(n) = \mathfrak{o}(n-2) \oplus \mathbb{R} f^n_{n-1} \oplus \mathrm{Span}(f^{n-1}_j + f^n_j, j = \overline{1, n-2})$.

Let $\mathfrak{m}$ be a subalgebra of $\mathfrak{o}_1(n)$ such that $\dim \mathfrak{m} \geq \frac{1}{2}(n-1)(n-2) + 1$, and let $\mathfrak{a} = \mathfrak{m} \cap \mathfrak{o}(n-1)$. Since $\mathfrak{o}(n-1)$ is a maximal subalgebra of $\mathfrak{o}(n)$, $\mathfrak{a}$ is a proper subalgebra of $\mathfrak{o}(n-1)$, and $\dim \mathfrak{a} \leq \dim \mathfrak{o}(n-2) = \frac{1}{2}(n-2)(n-3)$.

Moreover, since the natural map $\mathfrak{o}(n-1)/\mathfrak{a} \to \mathfrak{o}_1(n)/\mathfrak{m}$, given by $x \bmod \mathfrak{a} \to x \bmod \mathfrak{m}$, is one to one, it follows that

$$\dim \mathfrak{m} \leq \dim \mathfrak{o}_1(n) - \dim \mathfrak{o}(n-1) + \dim \mathfrak{a} \leq \tfrac{1}{2}(n-1)(n-2) + 1.$$

Then $\dim \mathfrak{m} = \frac{1}{2}(n-1)(n-2) + 1$ and $\dim \mathfrak{a} = \frac{1}{2}(n-2)(n-3)$, but a classical result [14] shows that if $n \neq 5$ and $\dim \mathfrak{a} = \dim \mathfrak{o}(n-2)$, then $\mathfrak{a}$ is conjugated to $\mathfrak{o}(n-2)$ in $\mathfrak{o}(n-1)$. Therefore, one may assume that, up to a conjugacy, $\mathfrak{m} = \mathfrak{o}(n-2) \oplus \mathfrak{p}$, where $\mathfrak{p}$ is a subspace of the orthocomplement $\mathfrak{b}$ of $\mathfrak{o}(n-2)$ in $\mathfrak{o}_1(n)$ w.r.t. the Killing form.

So $\mathfrak{m} = \mathfrak{o}(n-2) \oplus \mathfrak{p}$ is reductive decomposition and $\dim \mathfrak{p} = n-1$.

For each $j \leq n-1$, let $\mathfrak{b}_j = \mathbb{R} f^{n-1}_j \oplus \mathbb{R} f^n_j$. Then $\mathfrak{b} = \bigoplus_{j=1}^{n-1} \mathfrak{b}_j$.

Assume $n \geq 4$. For $i \neq j \leq n-2$, $\mathrm{ad}\, f^j_i$ maps $\mathfrak{b}_i$ isomorphically onto $\mathfrak{b}_j$, and leaves $\mathfrak{p}$ invariant; as such, all these subspaces $\mathfrak{p} \cap \mathfrak{b}_j$ are isomorphic. Moreover, they are of dimension 1 or less, since otherwise $\dim \mathfrak{p} \geq 2(n-1) > n-1$. Suppose $X \in \mathfrak{p}$ has the decomposition $X = X_1 + \cdots + X_{n-1}$ as an element of the direct sum $\mathfrak{b} = \bigoplus_{j=1}^{n-1} \mathfrak{b}_j$. We claim that all the components X_i are in $\mathfrak{p}$.

Indeed $\mathrm{ad}^2 f^j_i(X) = -X_i - X_j$, and therefore $X_1 + \cdots + X_{n-2}$ and X_{n-1} are in $\mathfrak{p}$. If $n = 4$, f^4_3 is in $\mathfrak{p}$ and by a straightforward computation one may prove that X_1 and X_2 are as well as in $\mathfrak{p}$. If $n > 4$ and $i, j, k < n-1$ are distinct indices, then $2X^i_i = (\mathrm{ad}^2 f^k_j - \mathrm{ad}^2 f^j_i - \mathrm{ad}^2 f^i_k)(X)$ so that all the components X_i are in $\mathfrak{p}$. It follows that $\mathfrak{p} = \bigoplus_{j=1}^{n-1}(p \cap \mathfrak{b}_j)$, and for any j, $\mathfrak{p} \cap \mathfrak{b}_j$ is one-dimensional, in particular $f^n_{n-1} \in \mathfrak{p}$.

Let $0 \neq f_j = a_j f^{n-1}_j + b_j f^n_j \in \mathfrak{b}_j \cap \mathfrak{p}$. Since $[f^n_{n-1}, f_j] \in \mathfrak{b}_j \cap \mathfrak{p}$ and $\dim \mathfrak{p} \cap \mathfrak{b}_j = 1$, it follows that $a^2_j = b^2_j$.

Suppose there is a subscript j such that $a_j = b_j$; then $f^{n-1}_j + f^n_j \in \mathfrak{b}_j \cap \mathfrak{p}$, and for any $k \neq j, k \leq n-2$, $[f^j_k, f^{n-1}_j + f^n_j] = f^{n-1}_k + f^n_k$ is a basis of $\mathfrak{b}_k \cap \mathfrak{p}$; thus $\mathfrak{m}$ is conjugated to $\mathfrak{m}(n)$.

Suppose there is some j such that $a_j + b_j = 0$. In this case $\mathfrak{m}$ is conjugated to $\mathfrak{m}_-(n) = \mathfrak{o}(n) \oplus \mathbb{R} f^n_{n-1} \oplus \mathrm{Span}(f^{n-1}_j - f^n_j, j = \overline{1, n-2})$, thus conjugated to $\mathfrak{m}(n)$ as well.

Assume $n = 3$. $\mathfrak{m}$ is a plane in $\mathfrak{o}$ (3); $\mathfrak{m} = \{\xi \in \mathfrak{o}(3), A\xi^1_2 + B\xi^1_3 + C\xi^2_3 = 0\}$. At least two of the coefficients A, B, C are nonzero. Notice that $A \neq 0$. Therefore, either $B = 0$ and $\mathfrak{m} = \mathrm{Span}(f^2_1 + f^3_2, f^3_1)$, or one may pick up a basis of $\mathfrak{m}$ of the form. $(X_1(b) = f^2_1 + bf^3_1, X_2(b) = \mp\sqrt{b^2 - 1} f^2_1 + bf^3_2)$; the subalgebras $\mathrm{Span}(f^2_1 + f^3_2, f^3_1)$, $\mathrm{Span}(X_1(b), X_2(b))$ are conjugated to $\mathfrak{m}$ (3). □

Now we specialize in $\mathfrak{o}_1$ (4) (see also [6, p. 39]).

In view of Proposition 2.1, $h(4) = 4$ and any four-dimensional Lie subalgebra of $\mathfrak{o}_1(4)$ is conjugated to $\mathfrak{m}(4)$. As far as the other proper subalgebras are concerned, in this dimension there are special techniques which allow us to find the conjugacy classes of *all Lie* subalgebras of $\mathfrak{o}_1(4)$. One of these is the fact that an element of $\mathfrak{o}_1(4)$ may be identified with a constant electromagnetic field. One associates with the electromagnetic field $F = (\mathbf{E}, \mathbf{H}) = (E^i e_i, H^j e_j)$ the element $f = E^1 f_1^4 + E^2 f_2^4 + E^3 f_3^4 + H^1 f_2^3 + H^2 f_3^1 + H^3 f_1^2$.

It is elementary [8] that a constant nonzero electromagnetic field may be reduced by means of a Lorentz transformation to one and only one of the following forms:

$$f_{E,H} = H f_1^2 + E f_3^4, \qquad f_0 = (f_1^2 + f_2^4). \tag{2.2}$$

We shall say that these are *canonical elements* of $\mathfrak{o}_1(4)$.

Corollary 2.1. A one-dimensional subalgebra of $\mathfrak{o}_1(4)$ is conjugated to one and only one of the subalgebras $\mathfrak{o}(2)$, $\mathfrak{o}_1(2)$, $\mathbb{R}f_0$, $\mathbb{R}f_{1,H}$, $H > 0$.

Remark 2.1. More generally, let $\langle X\rangle$ be the conjugacy class of the subalgebra $\mathbb{R}X$ of $\mathfrak{o}_1(n)$. Then if $[x]$ is the largest integer smaller than x, the following holds true.

If $n \geq 4$, $r_0 = [\frac{1}{2}n] - 2$, $r_+ = [\frac{1}{2}(n-1)] - 1$, $r_- = [\frac{1}{2}n] - 1$, then, for each nonzero $X \in \mathfrak{o}_1(n)$, there are unique $e \in \{0, +, -\}$, $a \in [0, \infty]^{r_e}$ such that $\langle X\rangle = \langle X_e(a)\rangle$, where

$$X_0(a) = \begin{cases} a_1 f_1^2 + \cdots + a_{r_0} f_{2r_0-1}^{2r_0} + f_{n-3}^{n-2} + f_{n-1}^{n} + f_n^{n+1}, & n \geq 5,\\ f_2^3 + f_3^4, & n = 4,\end{cases}$$
$$X_+(a) = a_1 f_1^2 + \cdots + a_{r_+} f_{2r_+-1}^{2r_+} + f_{n-2}^{n-1},$$
$$X_-(a) = a_1 f_1^2 + \cdots + a_{r_-} f_{2r_--1}^{2r_-} + f_{n-1}^{n}.$$

Another simplification comes from the existence of an action by conjugacy of $\mathrm{Sl}(2, \mathbb{C})$ on the space of 2×2 self-adjoint matrices of determinant 1, which yields a double covering morphism Φ of the component of the identity $\mathrm{O}_1(4)_0$ by $\mathrm{Sl}(2, \mathbb{C})$. The induced Lie algebra isomorphism is the inverse of the $\frac{1}{2}$-*spin representation* [8, p. 447] or [15, p. 44]. Let $\psi = \mathrm{d}_1\Phi^{-1} : \mathfrak{o}_1(4) \to \mathfrak{sl}(2, \mathbb{C})^{\mathbb{R}}$. Then

$$\begin{aligned} \psi(f_1^2) &= -\tfrac{1}{2}\mathrm{i}(E_1^1 - E_2^2), & \psi(f_1^3) &= \tfrac{1}{2}(E_2^1 - E_1^2),\\ \psi(f_1^4) &= \tfrac{1}{2}(E_2^1 + E_1^2), & \psi(f_2^3) &= -\tfrac{1}{2}\mathrm{i}(E_2^1 + E_1^2),\\ \psi(f_2^4) &= -\tfrac{1}{2}\mathrm{i}(E_2^1 - E_1^2), & \psi(f_3^4) &= \tfrac{1}{2}(E_1^1 - E_2^2).\end{aligned} \tag{2.3}$$

Two subalgebras of $\mathfrak{o}_1(4)$ are conjugated iff their ψ-images are conjugated in $\mathfrak{sl}(2, \mathbb{C})^{\mathbb{R}}$ under the adjoint action of $\mathrm{SL}(4, \mathbb{C})$ (or even of $\mathrm{GL}(2, \mathbb{C})$). We have to find the conjugacy classes of two- and three-dimensional real subalgebras of $\mathfrak{sl}(2, \mathbb{C})^{\mathbb{R}}$. But conjugated subalgebras are isomorphic, and two- and three-dimensional Lie algebras are classified since Bianchi. Therefore one may realize the abstract Lie algebras as subalgebras of $\mathfrak{sl}(2, \mathbb{C})^{\mathbb{R}}$.

Lemma 2.1. (1). A two-dimensional real Lie algebra is commutative or affine (has a basis (X_1, X_2) such that either $[X_1, X_2] = 0$, or $[X_1, X_2] = X_1$).

(2) A three-dimensional Lie algebra has a basis $X = (X_1, X_2, X_3)$, with respect to which, its structure equations have one and only one of the following forms (in each case, we write down only the nonzero brackets):

(0) commutative,

(1) $[X_2, X_3] = X_1$,

(2.a) $[X_1, X_2] = -aX_2, [X_1, X_3] = (a-2)X_3, a \geq 1$,

(2.b) $[X_1, X_2] = -X_2 + bX_3, [X_1, X_3] = -X_3 - bX_2, b > 0$,

(2.0) $[X_1, X_2] = -X_3, [X_1, X_3] = \mp X_2$,

(2) $[X_1, X_2] = -X_2 + X_3, [X_1, X_3] = -X_3$,

(3.+) $[X_1, X_2] = X_3, [X_1, X_3] = -X_2, [X_2, X_3] = X_1$,

(3.−) $[X_1, X_2] = -X_3, [X_1, X_3] = -X_2, [X_2, X_3] = X_1$.

In view of Lemma 2.1, one has to find the subalgebras of $\mathfrak{sl}(2, \mathbb{C})^{\mathbb{R}}$ of a given type, and among them to distinguish those that are conjugated. Moreover, w.l.o.g., one may assume that X_1 as a ψ-image of a canonical element. $\psi(2f_{E,H}) = (E - Hi)(E_1^1 - E_2^2) = Y_z$, and $\psi(f_0)$ is conjugated to $\mathrm{i}(E_1^1 - E_2^2 + f_1^2) = Y_0$.

2.1. Two-dimensional Lie subalgebras

Suppose $\mathfrak{h}$ is commutative, then $X_2 = aX_1$, $a \in \mathbb{C}\backslash\mathbb{R}$. If $X_1 = Y_z$, then $\mathrm{d}_1\phi(\mathfrak{h})$ is conjugated to $\mathfrak{o}(2) \oplus \mathfrak{o}_1(2) = \mathbb{R}f_1^2 \oplus \mathbb{R}f_3^4$. If $X_1 = Y_0$, then $\mathrm{d}_1\phi(\mathfrak{h})$ is conjugated to $\mathbb{R}(f_1^2 + f_2^4) \oplus \mathbb{R}(f_1^3 + f_3^4)$, which we denote by $\mathfrak{a}_2$. $\mathfrak{a}_2$ is not conjugated to $\mathfrak{o}(2) \oplus \mathfrak{o}_1(2)$.

Suppose $\mathfrak{h}$ is affine. Since the commutator of a diagonal matrix with another matrix has a zero diagonal, the possibility $X_1 = Y_z$ is excluded.

If $X_1 = Y_0$, then X_2 has the form $X_2 = \frac{1}{2}(1 - 2c)(E_1^1 - E_2^2) - cf_1^2 + E_1^2$.

If $2c = 1 - a + \mathrm{i}b$, it follows that $\mathfrak{h}$ is the Lie subalgebra $\mathfrak{h}_a$ given by

$$\mathfrak{h}_a = \{(au - \mathrm{i}v)(E_1^1 - E_2^2 + f_1^2) + u(E_2^1 + E_1^2), u, v \in \mathbb{R}\}.$$

A direct computation shows that if

$$g = 2I_2 + (a' - a)(E_1^1 - E_2^2 + f_1^2),$$

then $\operatorname{ad} g(\mathfrak{h}_a) = \mathfrak{h}_{a'}$, which proves that all two-dimensional affine subalgebras are conjugated. A representative of this conjugacy class in $\mathfrak{o}_1(4)$ is $\mathfrak{m}(3) = \mathbb{R}(f_2^3 + f_2^4) \oplus \mathbb{R}f_3^4$.

Proposition 2.2. Any two-dimensional Lie subalgebra of $\mathfrak{o}_1(4)$ is conjugated with one and only one of the following Lie subalgebras; $\mathfrak{a}_2$, $\mathfrak{o}(2) \oplus \mathfrak{o}_1(2)$, $\mathfrak{m}(3)$.

2.2. Three-dimensional Lie subalgebras

Y_0 cannot be the first element of a canonical basis like in Lemma (2.1) (2). Also, there are no Lie subalgebras of the type (0) or (1).

The only subalgebras of $\mathfrak{sl}(2,\mathbb{C})^{\mathbb{R}}$ of type (2.a) with $X_1 = Y_z$ are $\mathbb{R}(E_1^1 - E_2^2) \oplus \mathbb{R}E_1^2 \oplus \mathrm{i}\mathbb{R}E_1^2$ and $\mathbb{R}(E_1^1 - E_2^2) \oplus \mathbb{R}E_2^1 \oplus \mathrm{i}\mathbb{R}E_2^1$, which are conjugated to each other. The first one is the ψ-image of $\mathbb{R}(f_1^3 + f_1^4) \oplus \mathbb{R}(f_2^3 + f_2^4) \oplus \mathbb{R}f_3^4$ and we will denote this subalgebra by $\mathfrak{a}_{3,0}$.

For each $b > 0$, the subalgebras of type (2.b) are conjugated. $\mathbb{R}(f_1^3 + f_1^4) \oplus \mathbb{R}(f_2^3 + f_2^4) \oplus \mathbb{R}(f_3^4 - bf_1^2)$ is an element of this conjugacy class (in $\mathfrak{o}_1(4)$).

Now we are looking for subalgebras $\mathfrak{h}$ of $\mathfrak{sl}(2,\mathbb{C})^{\mathbb{R}}$ of type (2.0). Thus $\mathfrak{h} = \mathrm{Span}(X_1, X_2, X_3)$,

$$[X_1, X_3] = \mp X_2, \quad [X_1, X_2] = -X_3, \quad [X_2, X_3] = 0. \tag{2.4}$$

Without loss of generality, we may assume that $X_1 = z(E_1^1 - E_2^2)$, $z \in \mathbb{C}^*$, $X_j = a_j(E - E) + b_jE + c_jE$, $j = 2, 3$.

If in the first equation in (2.4) we select "$-$", it turns out that X_1, X_2, X_3 are linearly dependent.

If in the first equation of (2.4) we select "+", a direct computation shows that $\mathfrak{h} = \mathfrak{h}_1^2 = \{\mathrm{i}t(E_1^1 - E_2^2) + zE_1^2, t \in \mathbb{R}, z \in \mathbb{C}\}$, or $\mathfrak{h} = \mathfrak{h}_2^1 = \{\mathrm{i}t(E_1^1 - E_2^2) + zE_2^1, t \in \mathbb{R}, z \in \mathbb{C}\}$.

Both these algebras lie in the same conjugacy class of the ψ-image of $\mathfrak{a}_{3,1} = \mathbb{R}(f_1^3 + f_1^4) \oplus \mathbb{R}(f_2^3 + f_2^4) \oplus \mathbb{R}f_1^2$.

The conjugacy classes of Lie subalgebras of types (2.a), (2.b), and (2.0) may be thus parameterized by the closed interval $I = [0, 1]$; for each $t \in I$, let us put

$$\mathfrak{a}_{3,t} = \mathbb{R}(f_1^3 + f_1^4) \oplus \mathbb{R}(f_2^3 + f_2^4) \oplus \mathbb{R}((1-t)f_3^4 - tf_1^2). \tag{2.5}$$

As hyperplane of $\mathfrak{m}(4)$, $\mathfrak{a}_{3,t}$ are maximal Lie subalgebras. There are no Lie subalgebras of type (2). Any Lie subalgebra of type (3.+) is the Lie subalgebra of a maximal compact connected Lie subgroup of $\mathrm{O}_1(4)$, being thus conjugated to SO(3) [22].

The noncompact simple subalgebras (type (3.$-$)), with $X_1 = Y_z$, have the form $\mathfrak{h}_c = \{t(E_1^1 - E_2^2) + w(cE_1^2 - c^{-1}E_2^1), t \in \mathbb{R}, w \in \mathbb{C}\}$ with $c > 0$. If $a^2 = c'c$ and $g = aE_1^2 - a^{-1}E_2^1$, and then $\mathrm{ad}\, g(\mathfrak{h}_c) = \mathfrak{h}_{c'}$.

We proved:

Proposition 2.3. Any three-dimensional Lie subalgebra of $\mathfrak{o}_1(4)$ is conjugated with one and only one of the subalgebras $\mathfrak{o}(3)$, $\mathfrak{o}_1(3)$, or $\mathfrak{a}_{3,t}$, $t \in I$. Any maximal Lie subalgebra of $\mathfrak{o}_1(4)$ is conjugated with a Lie subalgebra of a Lie subgroup of $\mathrm{O}_1(4)$, leaving invariant a hyperplane of the four-dimensional Minkowski space.

3. Four-dimensional homogeneous Lorentz spaces with nontrivial isotropy via $\mathfrak{g}$-triples

The problem of listing four-dimensional locally homogeneous Lorentz manifolds amounts to the following algorithm [17]:

(a) List the conjugacy classes of Lie subalgebras of the Lorentz algebra $\mathfrak{o}_1(4)$.

(b) Given the complement $\mathfrak{p}$ of the subalgebra $\mathfrak{g}$ of $\mathfrak{o}_1(4)$, find the ad $\mathfrak{g}$-invariant linear maps $\Gamma : \mathbb{R}^4 \to \mathfrak{p}$ and the bilinear map $\Omega : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathfrak{g}$ whose associated AS – "torsion" and "curvature". $T : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^4$, $\tilde{\Omega} : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathfrak{g}$ defined by:

$$T(X, Y) = \Gamma(Y)X - \Gamma(X)Y, \tag{3.1}$$

$$\tilde{\Omega}(X, Y) = \bar{\Omega}(X, Y) - [\Gamma(X), \Gamma(Y)]_{\mathfrak{g}} \tag{3.2}$$

satisfy to:

$$\begin{aligned}
&[\xi, \tilde{\Omega}(X, Y)] - \tilde{\Omega}(\xi X, Y) - \tilde{\Omega}(X, \xi Y) \\
&\quad + [[\xi, \Gamma(X)]_{\mathfrak{g}}, \Gamma(Y)]_{\mathfrak{g}} + [\xi, \Gamma(T(X, Y)]_{\mathfrak{g}} \\
&\quad + [\Gamma(X), [\xi, \Gamma(Y)]_{\mathfrak{g}}]_{\mathfrak{g}} = 0 \\
&\quad \forall \xi \in \mathfrak{g}, \quad \forall X, \forall Y \in \mathbb{R}^4,
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
&\sum_{\mathrm{cl}} \tilde{\Omega}(T(X, Y), Z) - [\tilde{\Omega}(X, Y), \Gamma(Z)]_{\mathfrak{g}} = 0 \\
&\quad \forall X, \forall Y, \forall Z \in \mathbb{R}^4,
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
&\sum_{\mathrm{cycl}} (\tilde{\Omega}(X, Y)(Z) - T(T(X, Y), Z) = 0 \\
&\quad \forall X, \forall Y, \forall Z \in \mathbb{R}^4.
\end{aligned} \tag{3.5}$$

The ad $\mathfrak{g}$-invariance of Γ means

$$\Gamma(\xi X) = [\xi, \Gamma(X)]_{\mathfrak{p}} \quad \forall \xi \in \mathfrak{g}, \quad \forall X \in \mathbb{R}^4. \tag{3.6}$$

$A_{\mathfrak{g}}$ and $A_{\mathfrak{p}}$ stand for the components of A w.r.t. the decomposition $\mathfrak{o}_1(4) = \mathfrak{g} \oplus \mathfrak{p}$. Once the $\mathfrak{g}$-triple $(\mathfrak{p}, \Gamma, \bar{\Omega})$ is determined, one can associate with it a locally homogeneous Lorentz manifold, called the *geometric realization* of the $\mathfrak{g}$-triple $(\mathfrak{p}, \Gamma, \bar{\Omega})$, whose Lie algebra of Killing vector fields is isomorphic to $\mathfrak{h} = g \oplus \mathbb{R}^4, [\ ,\ ])$, where

$$\begin{aligned}
&[\xi, \eta] = [\xi, \eta], \qquad \xi, \eta \in \mathfrak{g}, \\
&[\xi, X] = \xi(X) + [\xi, \Gamma(X)]_{\mathfrak{g}}, \qquad \xi \in \mathfrak{g}, \quad X \in \mathbb{R}^4, \\
&[X, Y] = -T(X, Y) - \tilde{\Omega}(X, Y), \qquad X, Y \in \mathbb{R}^4.
\end{aligned} \tag{3.7}$$

The curvature tensor of the geometric realization $\Omega = \bar{\Omega} \oplus {}_{\mathfrak{p}}\Omega : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathfrak{o}_1(4)$, and has the $\mathfrak{p}$-component

$${}_{\mathfrak{p}}\Omega = [\Gamma(X), \Gamma(Y)]_{\mathfrak{p}} + \Gamma(T(X, Y)). \tag{3.8}$$

Note that the *Ricci form* is the bilinear symmetric form $\rho : \mathbb{R}^4 \times \mathbb{R}^4$, acting on an orthoframe e, by

$$\rho(e_i, e_j) = \mathrm{Tr}(x \to \Omega(x, e_i)e_j). \tag{3.9}$$

We apply the two steps algorithm shown above in order to recover from this perspective six-dimensional transitive Killing algebras on four-dimensional Lorentz manifolds, which is part of the list of "gravitational fields" exhibited in [19]. Besides, using formulas (3.8), it will follow that any such gravitational field is modeled on a symmetric space. At the end of the section we shall give few remarks concerning this list [19].

We shall look for $\mathfrak{g}$-triples, where $\mathfrak{g}$ is a subalgebra of $\mathfrak{o}_1(4)$, and $\dim \mathfrak{g} \geq 2$. We have seen in Section 1 that the list of representatives of conjugacy classes of d-dimensional Lie subalgebras of $\mathfrak{o}_1(4)$ is:

- $\mathfrak{m}(4)$ for $d = 4$,
- $\mathfrak{o}(3)$, $\mathfrak{o}_1(3)$ and $\mathfrak{a}_{3,\mathfrak{t}}$, $t \in [0, 1]$, for $d = 3$,
- $\mathfrak{o}((2)) \oplus \mathfrak{o}_1(2)$, $\mathfrak{m}(2)$ and $\mathfrak{a}_2$ for $d = 2$.

Thus, in the sequel, we take $\mathfrak{g}$ to be one of the above Lie subalgebras of $\mathfrak{o}_1(4)$. We indicate the results without giving all the details (the reader may either refind the results by himself, or look at classification done with other methods (e.g. [19]). What would not be standard for the method (that was already exemplified in [17]) will be explained.

We also mention without proof that if $\mathfrak{g}$ is one of these subalgebras, any $\mathfrak{g}$-triple is closed (see [17]), or in other words is associated to a homogeneous Lorentz space.

$\mathfrak{m}(4)$-*triples*. The geometric realization of such a triple is locally flat.

$\mathfrak{o}(3)$-*triples*. For the geometric realization $\mathbf{M}$, of such a triple, we have the alternatives:

(a) $\mathbf{M}$ has constant positive curvature (locally de-Sitter space),
(b) $\mathbf{M}$ is locally the product of a Euclidean line and a Lorentz manifold of constant curvature.

$\mathfrak{o}_{-1}(3)$-*triples*. For the geometric realization $\mathbf{M}$ of such a triple, we have the alternatives:

(a) $\mathbf{M}$ has constant negative curvature (locally anti-de-Sitter space–time),
(b) $\mathbf{M}$ is locally the product of a Riemannian manifold of constantcurvature with a Minkowski line.

$\mathfrak{a}_{3,\mathfrak{t}}$-triples, $t \in [0, 1]$. The geometric realization is locally flat.

$\mathfrak{a}_{3,1}$-triples. In order to identify the geometric realization of such a triple, we refer to the paper of a Cahen and Wallach [5]. We remind that a Lorentz manifold is *indecomposable* if, for any point $x \in \mathbf{M}$, the holonomy group Φ_x fixes only nontrivial isotropic substances of $T_x\mathbf{M}$. It is known [4] that any simply connected indecomposable symmetric space is either a space of constant curvature, or admits a solvable transitive Killing algebra, and there is some $\lambda \in \mathbf{S}^{n-3}$, $\lambda = (\lambda_i)_{i=\overline{1,n-2}}$ such that $\mathbf{M} = (\mathbb{R}^n - \mathbf{g}_\lambda)$, with

$$\mathbf{g}_\lambda(x, \mathrm{d}x) = \sum_{\mathrm{i}=1}^{n-2} (\mathrm{d}x^{\mathrm{i}})^2 - \mathrm{d}x^{n-1}\,\mathrm{d}x^n + \sum_{i=1}^{n-2} \lambda_{\mathrm{i}}(x^{\mathrm{i}})^2(\mathrm{d}x^{n-1})^2. \tag{3.10}$$

We will say that $\mathbf{M}$ is a $(+)$-space if all the components of λ are equal and positive, and a $(-)$-space if these components are equal and negative.

Proposition 3.1. If the geometric realization of an $\mathfrak{a}_{3,1}$-triple is not flat, then it is isometric to a $(+)$-space or to a $(-)$-space.

Proof. Let $\mathfrak{p} = \mathrm{Span}(f_1^4, f_2^4, f_3^4)$ be a complement of $\mathfrak{a}_{3,1}$ in $\mathfrak{o}_1(4)$. From (2.3)–(2.6), we get the following $\mathfrak{a}_{3,1}$-triple $(\mathfrak{p}, \Gamma, \bar{\Omega})$ form:

$$\begin{aligned}
&\Gamma = 0, \qquad \bar{\Omega}(e_1, e_2) = \bar{\Omega}(e_3, e_4) = 0,\\
&\bar{\Omega}(e_1, e_3) = \bar{\Omega}(e_1, e_4) = \alpha(f_1^3 + f_1^4),\\
&\bar{\Omega}(e_2, e_3) = \bar{\Omega}(e_2, e_4) = \alpha(f_2^3 + f_2^4).
\end{aligned} \tag{3.11}$$

From (3.3)–(3.6), one obtains the structure equations of a solvable transitive Killing algebra, $\mathfrak{k}$, of a symmetric space:

$$\begin{aligned}
&[e_1, e_2] = [e_3, e_4] = 0,\\
&[e_1, e_3] = [e_1, e_4] = -\alpha(f_1^3 + f_1^4),\\
&[e_2, e_3] = [e_2, e_4] = -\alpha(f_2^3 + f_2^4),\\
&[f_1^3 + f_1^4, e_1] = [f_2^3 + f_2^4, e_2] = -e_3 + e_4,\\
&[f_1^3 + f_1^4, e_2] = [f_2^3 + f_2^4, e_1] = 0,\\
&[f_1^3 + f_1^4, e_3] = [f_1^3 + f_1^4, e_4] = e_1,\\
&[f_2^3 + f_2^4, e_3] = [f_2^3 + f_2^4, e_4] = e_2,\\
&[f_1^3 + f_1^4, f_2^3 + f_2^4] = 0,\\
&[f_1^2, e_1] = -e_2, \qquad [f_1^2, e_2] = e_1,\\
&[f_1^2, e_3] = [f_1^2, e_4] = 0,\\
&[f_1^2, f_1^3 + f_1^4] = -(f_2^3 + f_2^4),\\
&[f_1^2, f_2^3 + f_2^4] = f_1^3 + f_1^4.
\end{aligned} \tag{3.12}$$

Let $\mathbf{K}$ be the simply connected Lie group of Lie algebra $\mathfrak{K}$, and let

$$\Theta = \sum_{i=1}^{4} \theta^i e_i \oplus \omega_2^1 f_1^2 \oplus \omega_3^1 (f_1^3 + f_1^4) \oplus \omega_3^2 (f_2^3 + f_2^4)$$

be the canonical form of $\mathbf{K}$.

Then (3.12) are equivalent to

$$\begin{aligned}
&\mathrm{d}\theta^1 + \omega_2^1 \wedge \theta^2 + \omega_3^1 \wedge (\theta^3 + \theta^4) = 0,\\
&\mathrm{d}\theta^2 + \theta^1 \wedge \omega_2^1 + \omega_3^2 \wedge (\theta^3 + \theta^4) = 0,\\
&\mathrm{d}\theta^4 = -\mathrm{d}\theta^3 = \theta^1 \wedge \omega_3^1 + \theta^2 \wedge \omega_3^2, \qquad \mathrm{d}\omega_2^1 = 0,\\
&\mathrm{d}\omega_3^1 + \omega_2^1 \wedge \omega_3^2 - \alpha\theta^1 \wedge (\theta^3 + \theta^4) = 0,\\
&\mathrm{d}\omega_3^2 - \omega_2^1 \wedge \omega_3^1 - \alpha\theta^2 \wedge (\theta^3 + \theta^4) = 0.
\end{aligned} \tag{3.13}$$

$\mathbf{K}$ is solvable and thus diffeomorphic to $\mathbb{R}^7$, and (3.13) has the global solution:

$$\begin{aligned}
&\theta^1 + \mathrm{i}\theta^2 = \exp(\mathrm{i}x)(\mathrm{d}u - v\,\mathrm{d}y),\\
&\omega_3^1 + \mathrm{i}\omega_3^2 = \exp(\mathrm{i}x)(\mathrm{d}v + \alpha u\,\mathrm{d}y),\\
&\theta^3 + \theta^4 = \mathrm{d}y, \omega_2^1 = \mathrm{d}x, \quad (t, x, y) \in \mathbb{R}^3, \quad (u, v) \in \mathbb{C}^2,\\
&2\theta^4 = \mathrm{d}t + (\alpha|u|^2 + |v|^2)\,\mathrm{d}y - (\bar{v}\,\mathrm{d}u + \bar{u}\,\mathrm{d}v).
\end{aligned}$$

The Pfaff system $\theta^1 = \theta^2 = \theta^3 = \theta^4 = 0$ has the prime integrals $u = x^1 + \mathrm{i}x^2$, $y = x^3$, $t = x^4$, and $\| \theta \|_1^2$ is projectable to $g = g_\lambda$ given by (3.10), with $\lambda = (\alpha, \alpha)$. If $\alpha \neq 0$, one makes a change to scale in x^4 to get the form of $\mathbf{g}$ given in Proposition 3.1. □

$\mathfrak{o}(2) \oplus \mathfrak{o}_1(2)$*triples*. The geometric realization of such a triple is locally the product of two surfaces of constant curvature, one Riemannian and the other Lorentz.

$\mathfrak{m}(3)$-triples. We have the following alternatives for the geometric realization $\mathbf{M}$ of such a triple:

(a) $\mathbf{M}$ has constant nonpositive curvature,

(b) $\mathbf{M}$ is locally the product of a Euclidean line and a Lorentz space of constant negative curvature.

$\mathfrak{a}_2$-*triples*. We shall prove the following result:

Proposition 3.2. The geometric realization of any $\mathfrak{a}_2$-triple is modeled by a symmetric space. There exists such Lorentz manifolds, that are not locally symmetric.

Proof. All the computations are done for the Lie subalgebra $\mathfrak{h}$, conjugated to $\mathfrak{a}_2$: $\mathfrak{h} = \mathrm{Span}(f_1^3 - f_1^4, f_2^3 - f_2^4)$. If we look for $\mathfrak{h}$-triples ($\mathfrak{p} = \mathrm{Span}(f_1^2, f_1^4, f_2^4, f_3^4), \Gamma, \bar{\Omega}$), after some computations, resulting from (3.3)–(3.6), one obtains:

$$\begin{aligned}
&\Gamma(e_1) = \Gamma(e_2) = 0, \qquad \Gamma(e_3) = cf_1^2 + df_3^4, \qquad \Gamma(e_4) = -cf_1^2 - df_3^4,\\
&\bar{\Omega}(e_1, e_2) = \bar{\Omega}(e_3, e_4) = 0,\\
&\bar{\Omega}(e_1, e_3) = \bar{\Omega}(e_1, e_4) = \alpha(f_1^3 - f_1^4) + \beta(f_2^3 - f_2^4),\\
&\bar{\Omega}(e_2, e_3) = -\bar{\Omega}(e_2, e_4) = \beta(f_1^3 - f_1^4) + \gamma(f_2^3 - f_2^4).
\end{aligned} \tag{3.14}$$

From (3.7), it follows that $\mathfrak{k} = (\mathfrak{h} \oplus \mathbb{R}^3, [\ ,\ ])$ is a solvable Lie algebra. If we denote $f_1^3 - f_1^4 = e_5$, $f_2^3 - f_2^4 = e_6$, then the structure equations of $\mathfrak{k}$ are (e_3 stands for $\bar{e}_3 = e_3 + e_4$):

$$\begin{aligned}
&[e_1, e_2] = [e_1, e_3] = [e_2, e_3] = 0,\\
&[e_1, e_4] = -ce_2 + \alpha e_5 + \beta e_6,\\
&[e_2, e_4] = ce_1 + \beta e_5 + \gamma e_6,\\
&[e_3, e_4] = de_3, \qquad [e_2, e_5] = 0,\\
&[e_1, e_5] = e_3, \qquad [e_1, e_6] = 0,\\
&[e_2, e_6] = e_3, \qquad [e_3, e_5] = 0,\\
&[e_3, e_6] = 0, \qquad [e_4, 4_5] = e_1 - de_5 + ce_6,\\
&[e_4, e_6] = e_2 - ce_5 - de_6, \qquad [e_5, e_6] = 0.
\end{aligned} \tag{3.15}$$

Moreover from (3.15), it follows that if $c = d = 0$, then $\mathfrak{k} = \mathfrak{h} \oplus \mathbb{R}^3$ is a reductive decomposition of a symmetric space ($[\mathbb{R}^3, \mathbb{R}^3] \subseteq \mathfrak{h}$), that we shall name an $\alpha\beta\gamma$-*space*. Notice that an $\alpha\beta\gamma$-space is generally indecomposable, cf. Remark 3.1 below.

It is obvious (use (3.14) and ${}_{\mathfrak{p}}\Omega = 0$) that for any values of the parameters $\alpha, \beta, \gamma, c, d$ the geometric realization $\mathbf{M}$ of the $\mathfrak{h}$-triple given by (3.14) is modeled on an $\alpha\beta\gamma$-space.

We claim that a $0\beta0$-space, $\beta \neq 0$, is an indecomposable symmetric space, that is not a Lie group with left invariant Lorentz metric (this is a consequence of Remark 3.1, but the reader may as well follow the argument given below that is independent of other reference). Assume the contrary; then, the Lie algebra $\mathfrak{k}$ given by (3.15), with $\alpha = \gamma = c = d = 0 \neq \beta$,

has a Lie subalgebra $\mathfrak{g} = \mathrm{Span}(e'_1, e'_2, e'_3, e'_4)$, where for each $i = \overline{1,4}$, $e'_i = e_i + A_i e_5 + B_i e_6$. Since $\mathfrak{g}$ is closed under $[\ ,\ ]$, and $[e'_1, e'_2] = (A_2 - B_1)e_3$ is in $\mathfrak{g}$, we have two possibilities:
(1) $e'_3 = e_3$, and in this case the $[\ ,\ ]$ closure condition implies that $X^2 + A = 0$, where

$$X = \begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix},$$

which fails since $\det A < 0$.
(2) $e'_3 \neq e_3$, and then $A_2 - B_1 = 0$. But $[e'_1, e'_3] = A_3 e_3$ and $[e'_2, e'_3] = B_3 e_3$. This imply $A_3 = B_3 = 0$, which is again impossible.

Consider now the $\mathfrak{g}$-triple given by (3.14), with $\alpha = \gamma = c = 0$; we shall show that for $|d|$ large enough, $\mathfrak{k}$ has a four-dimensional Lie subalgebra, transverse to $\mathfrak{g}$. Indeed, let $\mathfrak{g} = \mathrm{Span}(e'_1, e'_2, e_3, e_4)$, where as in the previous case $e'_i = e_i + A_i e_5 + B_i e_6$, $i = 1, 2$. Then, the condition $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}$ is equivalent to $X^2 + dX + A = 0$, where X and A have the same meaning as in the previous case. If $d = 2m$, then $\mathfrak{g}$ is a subalgebra of $\mathfrak{k}$ iff the equation $(X + mI_2)^2 = -A + m^2 I_2$ has a solution in $M_2(\mathbb{R})$. This happens iff $|\beta| \leq m^2$. □

As a consequence of this computational analysis, we obtain:

Theorem 3.1. *Any Lorentz manifold of dimension 4, that admits a transitive Killing algebra of dimension ≥ 6, is modeled on a symmetric space. There are homogeneous Lorentz manifolds modeled on symmetric spaces which are not symmetric.*

We close this section with the two observations:

Remark 3.1. In order to see whether an $\alpha\beta\gamma$-space is indecomposable or not, we recall that for a reductive decomposition $\mathfrak{k} = \mathfrak{g} \oplus \mathbb{R}^4$, linear isotropy representation [11, Vol. 2] of the holonomy algebra at a point is the Lie algebra generated by $\{[X, Y]_{\mathfrak{h}}, X, Y \in \mathbb{R}^n\}$ [11, Vol. 2, p. 206]. In our case this Lie subalgebra of $\mathfrak{o}_1(4)$ may be one of the following: $\mathbb{R}(f_1^3 - f_1^4 + f_2^3 - f_2^4)$, $\mathbb{R}(f_1^3 - f_1^4)$, $\mathbb{R}(f_2^3 - f_2^4)$ or $\mathfrak{g}$.

Out of these four subalgebras of $\mathfrak{o}_1(4)$, $\mathfrak{g}$ leaves invariant only isotropic subspaces of the Minkowski vector space. But $\lambda_x \Phi_x = \mathfrak{g}$ iff $\alpha\gamma - \beta^2 \neq 0$. Consequently the only $\alpha\beta\gamma$-spaces that are indecomposable are those with $\alpha\gamma - \beta^2 \neq 0$.

Remark 3.2. The $\mathfrak{a}_2$-triples correspond to the family (33.53) on the list of gravitational fields presented in [19]. This is the only nontrivial example of four-dimensional Lorentz algebra that admits a six-dimensional transitive Killing algebra. This follows from the upper list $\mathfrak{g}$-triples. Notice that, unlike stated in [19], the Killing algebras (33.44), (33.45) are not transitive.

If one wishes to find the metric of a geometric realization of an arbitrary $\mathfrak{a}_2$-triple, one encounters in [19] a difficult problem of integration of system of differential equations (33.56). From our point of view, this problem is equivalent to the discussion of the nature of the characteristic roots of the matrix $E \in M_4\mathbb{R}$, given below:

$$E = -\alpha E_3^1 - \beta(E_4^1 + E_3^2) - \gamma E_4^2 + c(E_2^1 - E_1^2 + E_4^3 - E_3^4) - d(E_3^3 + E_4^4).$$

Although the characteristic equation has the elementary form $u\lambda^4 + v\lambda^2 + w = 0$, where u, v, w are polynomials in α, β, γ, c, d. A complete discussion of this equation has not yet been done.

We also leave to the curious reader the task of listing homogeneous spaces with a five-dimensional transitive Killing algebra, an instructive application of the method of $\mathfrak{g}$-triples, which is too long to be presented here.

4. Some examples of Ricci curvature homogeneous Lorentz manifolds

We will exhibit nonhomogeneous examples of R.c.h.'s along the following lines, which parallel a similar construction in Riemannian geometry [12]: Consider a one parameter (say x^1) group of isometries ϕ^A-acting on the Minkowski space $\mathbb{R}^3_1$, generated the matrix $\mathbf{A} \in \mathfrak{o}_1(3)$. Our Lie group $\mathbf{M}$ is generated by translations and by ϕ^A in the Poincaré group of $\mathbb{R}^3_1$. A point in $\mathbb{R}^3_1$ has the coordinates $(x^a, a = 2, 3, 4)$; in these coordinates, the Cartan–Maurer form of $\mathbf{M}$ is $\underline{\Theta} = \underline{\theta}^i E_i$, where

$$\underline{\theta}^1 = \mathrm{d}x^1, \qquad \underline{\theta}^a = \mathrm{d}x^a + A^a_b x^b \,\mathrm{d}x^1. \tag{4.1}$$

The structure equations of this group are

$$\mathrm{d}\underline{\theta}^1 = 0, \qquad \mathrm{d}\underline{\theta}^a = A^a_b \underline{\theta}^b \wedge \underline{\theta}^1. \tag{4.2}$$

$\underline{\Theta}$ can be also regarded as an orthocoframe for the left invariant flat metric $\mathbf{g}_0 = \| \underline{\Theta} \|^2_1$.

Let $\Theta \in \Omega^1(\mathbf{M}, \mathbb{R}^4)$ be a perturbation of $\underline{\theta}$ whose components are

$$\theta^1 = f(x^a)\underline{\theta}^1, \quad \theta^a = \underline{\theta}^a, \tag{4.3}$$

where f is a differentiable function on $\mathbf{M}$, and let $\mathbf{g}$ denote the metric $\| \Theta \|^2_1$; we differentiate (4.3) and based on (4.2) we get

$$\mathrm{d}\theta^1 = -\frac{\partial f}{\partial x^a} f^{-1}\theta^1 \wedge \theta^a, \qquad \mathrm{d}\theta^a = -f^{-1}A^a_b\theta^1 \wedge \theta^b. \tag{4.4}$$

Then a standard computation yields the following Levi-Civita connection forms w.r.t. Θ:

$$\omega^1_a = -\frac{\partial f}{\partial x^a} f^{-1}\theta^1, \qquad \omega^a_b = -f^{-1}A^a_b\theta^1, \tag{4.5}$$

and curvature forms

$$\Omega^1_a = -\frac{\partial^2 f}{\partial x^a \partial x^b} f^{-1}\theta^1 \wedge \theta^b, \qquad \Omega^a_b = 0. \tag{4.6}$$

The Ricci curvature of the perturbed metric w.r.t. Θ is

$$\begin{aligned} \rho_{11} &= -\sum{}_1 \delta^a_a \frac{\partial^2 f}{(\partial x^a)^2} f^{-1}, \\ \rho_{ab} &= -\frac{\partial^2 f}{\partial x^a \partial x^b} f^{-1}, \qquad \rho_{1a} = -\sum \frac{\partial^2 f}{\partial x^a \partial x^b} f^{-1}. \end{aligned} \tag{4.7}$$

It follows that $(\mathbf{M}, \mathbf{g})$ is R.c.h. iff there is a symmetric matrix (λ_{ab}) such that f is a solution for the linear system of p.d.e. s

$$\frac{\partial^2 f}{\partial x^a \partial x^b} = \lambda_{ab} f. \tag{4.8}$$

If we differentiate (4.6) once more, from the structure equations we get the following identities:

$$\lambda_{ab} \frac{\partial f}{\partial x^c} \theta^1 \wedge \theta^b \wedge \theta^c = 0,$$

and by a variant of Cartan's lemma (for Cartan's lemma see [9]), it follows that there are constants μ_a, ν_a such that

$$\lambda_{ab}\theta^b = \mu_a \theta^1 + \nu_a \frac{\partial f}{\partial x^c} \theta^c. \tag{4.9}$$

If we evaluate (4.9) on the orthoframe dual to Θ, we obtain the identity

$$\lambda_{ac} = \nu_a \frac{\partial f}{\partial x^c},$$

which proves that the matrix (λ_{ab}) has rank 1. The entries of this symmetric matrix have to have the form

$$\lambda_{ab} = c_a c_b \tag{4.10}$$

for some constant vector $\mathbf{c} = (c_a)$.

One may solve the system (4.8) whose general local solution is

$$f(x) = g(x^1)\exp(c_b x^b). \tag{4.11}$$

We proved:

Proposition 4.1. For any function f defined in (4.10), $(\mathbf{M}, \mathbf{g})$ is a R.c.h. Lorentz manifold. Moreover all these manifolds are c.h.

A Lorentz manifold $(\mathbf{M}, \mathbf{g})$ (or a metric $\mathbf{g}$) has a homogeneous model if there is a homogeneous Lorentz manifold $(\mathbf{H}, \mathbf{g}')$ with same curvature tensor as $\mathbf{M}$. For example if in (4.10) $\mathbf{c}$ has the form $(0, a, a)$, then $(\mathbf{M}, \mathbf{g})$ defined in (4.11) has for homogeneous model the $\alpha 00$-symmetric space, whose Killing algebra has the structure equations (3.15), with $c = d = 0$ in Section 1. In general, c.h. spaces are not locally homogeneous, or even do not have a homogeneous model [12].

In order to prove the next claim, we need a technical result.

Lemma 4.1. Assume $(\mathbf{E}_a)$ is an orthoframe field in a pseudo-Riemannian manifold $(\mathbf{M}, \mathbf{g})$ with $\varepsilon^a = \mathbf{g}(\mathrm{E_a}, \mathrm{E_a})$. Let $\| \ \|_{\mathrm{g}}^2$ be the "square" of the induced pseudo-Riemannian metric on the algebra of covariant tensor fields of $\mathbf{M}$ and let $\mathbf{t}$ be covariant tensor fields on $\mathbf{M}$. Then

$$\| \mathbf{t} \|_g^2 = \sum \varepsilon^a \| \mathrm{int}_{E_a} \mathbf{t} \|_g^2 .$$

Proof. Use the identity $\mathbf{t} = \sum \theta^a \otimes \mathrm{int}_{E_a}\mathbf{t}$ and the fact that on a decomposable tensor field, $\| \ \|_g^2$ acts multiplicatively, i.e. $\| \mathbf{s} \otimes \mathbf{u} \|_g^2 = \| \mathbf{s} \|_g^2 \| \mathbf{u} \|_g^2$. □

Proposition 4.2. The spaces $(\mathbf{M}, \mathbf{g})$ given by (4.11) are not locally homogeneous if $\mathbf{c}$ is not a null vector and $\| A\mathbf{c} \|_1^2$ does not vanish.

Proof. The covariant differential of the Ricci tensor of a locally homogeneous Lorentz manifold has constant induced "square" norm.

A direct computation shows that, in terms of the section Θ defined in (4.3), the Ricci tensor field is

$$\rho = \| \mathbf{c} \|_1^2 \, \theta^1 \otimes \theta^1 + \tfrac{1}{2} c_a c_b (\theta^a \otimes \theta^b + \theta^b \otimes \theta^a). \tag{4.12}$$

If $E = (E_i)$ is the orthoframe field dual to Θ, then from (4.5) we get

$$\nabla_X E_1 = \frac{\partial f}{\partial x^a} f^{-1} \theta^1(X) E_a,$$
$$\nabla_X E_a = f^{-1} A_a^b \theta^1(X) E_b - \frac{\partial f}{\partial x^a} f^{-1} \theta^1(X) E_1,$$

and therefore

$$\begin{aligned} \nabla_x \theta^a &= -\varepsilon_a \frac{\partial f}{\partial x^a} f^{-1} \theta^1(X) \theta^1 - f^{-1} A_c^a \theta^1(X) \theta^c, \\ \nabla_X \theta^1 &= \frac{\partial f}{\partial x^b} f^{-1} \theta^1(X) \theta^b. \end{aligned} \tag{4.13}$$

We take then covariant derivative of (4.12) w.r.t. X and from (4.13) and due to (4.11) we get

$$\begin{aligned} \nabla_x \rho = {} & \theta^1(X)((- \| \mathbf{c} \|_1^2 \, c_b (\theta^1 \otimes \theta^b + \theta^b \otimes \theta^1)) \\ & - 2 c_a c_b f^{-1} A_d^a (\theta^d \otimes \theta^a + \theta^a \otimes \theta^d)). \end{aligned} \tag{4.14}$$

In particular, the covariant derivatives of the Ricci tensor w.r.t. Θ are

$$\begin{aligned} \nabla_{E_1} \rho = {} & - \| \mathbf{c} \|_1^2 \, c_b (\theta^1 \otimes \theta^b + \theta^b \otimes \theta^1) \\ & - 2 c_a c_b f^{-1} A_d^a (\theta^d \otimes \theta^a + \theta^a + \theta^a \otimes \theta^d) \end{aligned} \tag{4.15}$$

and

$$\nabla_{E_a} \rho = -2 c_a c_b f^{-1} A_d^b (\theta^d \otimes \theta^a + \theta^a \otimes \theta^d). \tag{4.16}$$

Since $\mathrm{int}_{E_a} \nabla \rho = \nabla_{E_a} \rho$, in view of Lemma 4.1 it follows that the "square" norm of the covariant differential of the Ricci tensor is

$$\| \nabla \rho \|^2 = \| \nabla_{E_1} \rho \|^2 + \sum \varepsilon^a \| \nabla_{E_a} \rho \|^2 . \tag{4.17}$$

From (4.15) and (4.16) we get

$$\| \nabla_{E_1} \rho \|^2 = 4 \| \mathbf{c} \|_1^2 \, (\| \mathbf{c} \|_1^2 + 2 f^{-2} \| A\mathbf{c} \|_1^2)$$

and

$$\| \nabla_{E_a} \rho \|^2 = 8 f^{-2} \| \mathbf{c} \|_1^2 \| A\mathbf{c} \|_1^2 .$$

Thus the "square norm" of the covariant differential of the Ricci tensor is

$$\| \nabla \rho \|^2 = 4 \| \mathbf{c} \|_1^2 (\| \mathbf{c} \|_1^2 + 4 f^{-2} \| A\mathbf{c} \|_1^2). \tag{4.18}$$

Note that the right-hand side of (4.18) depends on the nonconstant function f and therefore $\| \nabla \rho \|$ is constant only if $\| \mathbf{c} \|_1^2 \| A\mathbf{c} \|_1^2$ vanishes, proving the claim. □

We recall that a necessary condition in one of Hawking singularity theorems is time-like convergence condition, i.e. $\mathbf{Ric}(\mathbf{v}, \mathbf{v}) \geq 0$ for any time-like vector $\mathbf{v}$.

Proposition 4.3. If $\mathbf{c}$ is a space-like vector, then the space $(\mathbf{M}, \mathbf{g})$ given by (4.11) satisfies the time convergence condition.

The proof is straightforward from the definition and (4.12).

A similar class of examples of c.h. Lorentz manifolds sources from an analogous construction, if we start instead with a skew symmetric matrix $\mathbf{B} \in \mathfrak{o}(3)$, and with a perturbation of the flat left invariant Lorentz metric on the group of motions $\mathbf{M}$ in the 3-Euclidean space, which are generated by translations and the 1-parameter group generated by $\mathbf{B}$. In this case the coordinates are x^a, $a = 1, 2, 3$, for the translation in the direction of x^a-coordinate axis and t for the 1-parameter group of isometires generated by $\mathbf{B}$. The left invariant forms on $\mathbf{M}$ are

$$\underline{\theta}^a = \mathrm{d}x^a + B^a_b x^b \,\mathrm{d}t, \qquad \underline{\theta}^4 = \mathrm{d}t \tag{4.19}$$

and if $\underline{\Theta} = (\underline{\theta}^i)$, then the flat left invariant Lorentz metric is $\mathbf{g}_0 = \| \underline{\Theta} \|_1^2$ and the perturbed metric is $\mathbf{g} = \| \Theta \|_1^2$, where the components of Θ are

$$\theta^a = \underline{\theta}^a, \quad \theta^4 = f \underline{\theta}^4 \tag{4.20}$$

and

$$f(x, t) = g(t) \exp(c_b x^b). \tag{4.21}$$

$(\mathbf{M}, \mathbf{g})$ has the curvature forms:

$$\Omega^a_b = 0, \qquad \Omega^4_a = c_a c_b \theta^4 \wedge \theta^b. \tag{4.22}$$

Remark 4.1. Note that the idea of perturbing a metric with additional properties, to obtain exact solutions of Einstein equations goes back to the Kerr-Schild solution, obtained by a perturbation from a flat metric and has been also used in a slightly more general setup in [21,23]. Interesting enough, if a metric (4.3) is R.c.h., then it is also c.h.; therefore examples of R.c.h.'s other than Einstein or c.h. in dimension 4 are still to be found.

Some of the early relativistic cosmological models like those due to Einstein, de Sitter, Gödel and others [10], being static had to be replaced by Robertson–Walker universes, or more general by space-like homogeneous Lorentz manifolds.

The main evidence of expansion of the Universe are the cosmological redshifts [10]. We will prove that such redshifts are predictable for a class of c.h. Lorentz manifolds with a metric (4.23).

Let us consider the metric $\mathbf{g}_{c,h} = \| \Theta \|_1^2$, where

$$\theta^a = \mathrm{d}x^a + h^a(t)\,\mathrm{d}t, \qquad \theta^4 = \exp(c_b x^b)\,\mathrm{d}t. \tag{4.23}$$

Theorem 4.1. *Assume h is an arbitrary differentiable function and $\| \mathbf{c} \| \neq 0$. Then a gravitational field modeled on a metric $\mathbf{g}_{c,h}$ has both light-like geodesics presenting both cosmological blueshifts and redshifts. Such a geodesic has a singularity, if it presents a blueshift, and lasts indefinitely in future if it presents a redshift.*

Proof. We will assume that s is an affine parameter along the light-like geodesic γ, and let $\dot{\gamma}$ be the derivative w.r.t. s. We put $\dot{\Theta} = \Theta(\dot{\gamma})$, so that the energy of a photon emitted along γ when measured w.r.t. Θ is $\dot{\Theta}^4$ and the momentum is $(\dot{\Theta}^a)$. We have $(\dot{\Theta})^2 = \sum(\dot{\Theta}^a)^2$, and therefore

$$\frac{\mathrm{d}}{\mathrm{d}s}(\dot{\Theta}^4)^2 = \sum \frac{\mathrm{d}}{\mathrm{d}s}(\dot{\Theta}^a)^2 \tag{4.24}$$

The Lagrangian is $\mathbf{L}(\gamma, \dot{\gamma}) = \sum(\dot{\Theta}^a)^2 - (\dot{\Theta}^4)^2$, where

$$\dot{\Theta}^a = \dot{x}^a + h^a(t)\dot{t}, \qquad \dot{\Theta}^4 = \exp(c_b x^b)\dot{t}.$$

The Euler–Lagrange equation for the ath component of γ is

$$2\frac{\mathrm{d}}{\mathrm{d}s}\dot{\Theta}^a = -2\dot{\Theta}^4 c_a \exp(c_b x^b)\dot{t}. \tag{4.25}$$

Multiply this equation by $\dot{\Theta}^a$ and sum up for a =1–3; it follows that

$$\frac{\mathrm{d}}{\mathrm{d}s}(\| \dot{\Theta}_{\text{space}} \|^2) = -2\dot{\Theta}^4 \sum_a c_a \dot{\Theta}^a \exp(c_b x^b)\dot{t} = -2\sum_a c_a \dot{\Theta}^a \exp(2c_b x^b)\dot{t}^2.$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}s}(\dot{\Theta}^4)^2 = -2(\dot{\Theta}^4)^2 \left(\sum_a c_a \dot{\Theta}^a\right).$$

Then multiply (4.24) by c_a and sum up for $a = 1$–3; it follows that

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\sum c_a \dot{\Theta}^a\right) = - \| c \|^2 \exp(2c_b x^b)\dot{t}^2 = - \| c \|^2 (\dot{\Theta}^4)^2.$$

Let $u = (\sum_a c_a \dot{\Theta}^a)$, $v = (\dot{\Theta}^4)^2$, $\alpha = \| c \|$. We obtained the o.d.e.:

$$\frac{\mathrm{d}}{\mathrm{d}s}u = -\alpha^2 v, \qquad \frac{\mathrm{d}}{\mathrm{d}s}v = -2uv,$$

and are led to consider the following o.d.e.:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}}{\mathrm{d}s}\ln v\right) = -2\alpha^2 v.$$

Set $v = \exp z$, then

$$\ddot{z} = -2\alpha^2 \exp z$$

and if we set $\dot{z} = p(z)$, then

$$p'p = -2\alpha^2 \exp z,$$

which implies

$$\dot{z}^2 = p^2 = 4\alpha^2(\exp c - \exp z)$$

so that $\dot{z} = \pm 2\alpha(\exp c - \exp z)^{1/2}$, where c is a constant.

Assume $z = 2\alpha w$, and let $k = \exp c - \exp 2\alpha$, then $\dot{w} = \pm(k - \exp w)^{1/2}$, where k is a positive constant.

Case 1. $\dot{w} = -(k - \exp w)^{1/2}$. w decreases and so does $\exp w$, and therefore $k - \exp w$ increases and remains negative. Assume $s_0 < s$ and $W = W(s)$, $W_0 = \exp s_0$. Then

$$s - s_0 = \int_W^{W_0} (k - \exp w)^{1/2}\, \mathrm{d}w$$

and

$$\lim_{W\to-\infty} \int_W^{W_0} (k - \exp w)^{1/2}\, \mathrm{d}w = \infty$$

showing that the light-like geodesic extends indefinitely in the future.

Case 2. $\dot{w} = (k - \exp w)^{1/2}$. w increases, $k - \exp w$ decreases. Question is how long will it take to w, to reach the value $\ln k$. Let $s_0 < s$ and $W = W(s)$, $W_0 = \exp s_0$. One has to find the limit

$$s_k = s_0 + \lim_{W\to\ln k} \int_{W_0}^{W} (k - \exp w)^{1/2}\, \mathrm{d}w.$$

This integral is convergent and therefore in Case 2, the light-like geodesic has a singularity in the future.

From (4.23), the physical interpretation of this behavior of light-like signals is that they are either inextensible, or else $E(\gamma(s)) = (\dot{\Theta}^4)^2$ decreases. This energy $E(\gamma(s))$ is $h/\lambda(s)$, where h is Planck's constant and $\lambda(s)$ is the wavelength of our "photon". measured in the coframe Θ. As such, in the second case, λ is increasing in time along the geodesic, and the photon has a positive redshift. In the first case, one should expect blueshift of light, and since the geodesic is inextendible, the physical interpretation is that blueshift signals a singularity. □

Corollary 4.1. Assume $(\mathbf{M}, \mathbf{g})$ is a c.h. Universe given by (4.19), with $\| \mathbf{c} \| \neq 0$ and $\mathbf{B} = \mathbf{0}$. The wavelength of any light-like particle which lives indefinitely in the future presents a redshift.

Proposition 4.4. The metrics in Theorem 4.1 satisfy the time divergence condition, i.e. for any time-like vector $\mathbf{v}$, $\mathbf{Ric}(\mathbf{v}, \mathbf{v}) \leq \mathbf{0}$.

Proof. A straightforward computation following from (4.22) shows that the Ricci tensor field is

$$\mathbf{Ric} = - \| \mathbf{c} \|^2 \theta^4 \otimes \theta^4 + \tfrac{1}{2}\mathbf{c}_a \mathbf{c}_b (\theta^a \otimes \theta^b + \theta^b \otimes \theta^a).$$

Thus if $v = v^i E_i$ is a time-like vector,

$$\begin{aligned}\mathbf{Ric}(\mathbf{v}, \mathbf{v}) &= \langle \mathbf{v}_{\text{space}}, \mathbf{c} \rangle^2 - (\mathbf{v}^4)^2 \| \mathbf{c} \|^2 \leq \| \mathbf{v}_{\text{space}} \|^2 \| \mathbf{c} \|^2 - (\mathbf{v}^4)^2 \| \mathbf{c} \|^2 \\ &= \mathbf{g}(\mathbf{v}, \mathbf{v}) \| \mathbf{c} \|^2 \leq \mathbf{0}. \qquad \square\end{aligned}$$

Remark 4.2. From Proposition 4.3 it follows that the metrics in Theorem 4.1 are typical "antimatter" gravitational fields, since the gravity in these space–times is repelling.

Thus the singularity theorems related to these metrics is *not* a formal consequence of Hawking's theorem.

We finally note that existence of singularities leads us to the following question: Find sufficient conditions, under which a space–time (4.11), (4.20) or more generally a space–time of constant energy–momentum tensor is globally hyperbolic, or satisfies the generic condition [1,10].

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